Research Proposal for PhD: Combinatorial Optimization and Advanced Mathematical Frameworks

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Introduction

Operations Research (OR) is an intersection of mathematics, engineering, and management science, dedicated to optimizing complex decision-making processes. I am incredibly interested to the scope of problems Operations Research can solve using combinatorial optimization and its integration with probability theory, functional analysis and measure theory. In this document, I will shed light over my research interests and present my research proposal for a five-year PhD program.

Research Objectives

In this section, I will list the areas **(according to their importance and relevance to my interests)** and give a short overview of the topics.

1. **Combinatorial Optimization in High-Dimensional and Dynamic Spaces**
    - Investigate polyhedral combinatorics and facet analysis to study and improve integer programming formulations for **NP-hard** problems. For example: comb analysis, stronger cuts, VRP with time windows, capacitated VRPWT, etc.
    - Develop scalable algorithms using graph-theoretic methods for dynamic resource allocation in evolving networks. For eg: Bipartite matching, network flow optimization, etc.
2. **Integration of Probability Theory in Stochastic Optimization**
    - Formulate and analyze multistage stochastic programming models using measure-theoretic probability frameworks.
    - Extend Markov Decision Processes (MDPs) to accommodate combinatorial state spaces with rigorous probabilistic underpinnings.
3. **Application of Functional Analysis and Measure Theory in Optimization Algorithms**
    - Utilize Banach and Hilbert space theories to study the convergence properties of iterative optimization algorithms.

- Apply measure-theoretic integration techniques to decompose and solve high-dimensional optimization problems.

Methodology

1. Combinatorial Optimization in High-Dimensional and Dynamic Spaces

Combinatorial optimization seeks optimal solutions from discrete, finite sets, which becomes increasingly complex in high-dimensional spaces.

1.1. Polyhedral Combinatorics and Facet Analysis

Investigate the polyhedral structures of combinatorial problems such as the Traveling Salesman Problem (TSP) and Integer Linear Programming (ILP). By analyzing the facets of these polyhedra, we can derive tighter linear relaxations, improving the performance of branch-and-cut algorithms.

$$\text{Minimize} \quad \mathbf{c}^\top \mathbf{x} \quad \text{subject to} \quad \mathbf{A}\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \in \mathbb{Z}^n$$

where:

- $\mathbf{c} \in \mathbb{R}^n$ is the cost vector,
- $\mathbf{x} \in \mathbb{Z}^n$ is the vector of decision variables,
- $\mathbf{A} \in \mathbb{R}^{m \times n}$ is the constraint matrix,
- $\mathbf{b} \in \mathbb{R}^m$ is the constraint bounds.

1.2. Graph-Theoretic Methods for Dynamic Resource Allocation

Employ graph-theoretic constructs such as network flows, shortest path algorithms, and matching theory to model and solve dynamic resource allocation problems. The dynamic nature is captured through time-expanded networks or adaptive graph models.

$$\text{Maximize} \quad \sum_{(i,j)\in E} f_{ij} \quad \text{subject to} \quad \sum_{j:(i,j)\in E} f_{ij} - \sum_{j:(j,i)\in E} f_{ji} = b_i, \quad \forall i \in V$$

Where:

- $G = (V, E)$: A graph with vertices V and edges E.
- f_{ij}: Flow on the edge $(i, j) \in E$.
- b_i: Supply/demand at node i:
  - $b_i > 0$: Supply at node i.
  - $b_i < 0$: Demand at node i.
  - $b_i = 0$: Transshipment at node i.
- $\sum_{j:(i,j)\in E} f_{ij}$: Total outgoing flow from node i.
- $\sum_{j:(j,i)\in E} f_{ji}$: Total incoming flow to node i.

2. Integration of Probability Theory in Stochastic Optimization

Incorporating probabilistic elements into optimization models ensures robustness against uncertainty.

2.1. Multistage Stochastic Programming with Measure-Theoretic Foundations

Develop multistage stochastic programming models utilizing measure-theoretic probability to accurately represent scenario trees and uncertainty sets.

$$\min_{\mathbf{x}} \mathbb{E}_{\mathbb{P}} \left[f(\mathbf{x}, \xi) \right]$$

where $\mathbb{P}$ is a probability measure on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

2.2. Combinatorial Markov Decision Processes

Extend MDPs to combinatorial state spaces by integrating measure-theoretic stochastic processes, ensuring well-defined transition probabilities and reward distributions.

$$\mathcal{M} = (\mathcal{S}, \mathcal{A}, P, R, \gamma)$$

where $\mathcal{S}$ is a combinatorial state space, $\mathcal{A}$ is the action space, $P : \mathcal{S} \times \mathcal{A} \to \mathcal{P}(\mathcal{S})$ is the transition probability kernel, $R : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ is the reward function, and $\gamma \in (0, 1)$ is the discount factor.

3. Functional Analysis and Measure Theory in Optimization Algorithms

Functional analysis provides the tools to study the properties of infinite-dimensional spaces and operators, which are essential for analyzing optimization algorithms.

3.1. Banach and Hilbert Space Frameworks

Formulate optimization problems within Banach and Hilbert spaces to leverage fixed-point theorems and spectral analysis for algorithm convergence.

$$\text{Find } x \in H \quad \text{such that} \quad T(x) = x$$

where H is a Hilbert space and $T : H \to H$ is a bounded linear operator.

3.2. Measure-Theoretic Integration Techniques

Apply Lebesgue integration and Fubini's theorem to decompose high-dimensional optimization problems into tractable subproblems.

$$\int_{X \times Y} f(x, y) \, d(\mu \times \nu)(x, y) = \int_X \left(\int_Y f(x, y) \, d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X f(x, y) \, d\mu(x) \right) d\nu(y)$$

This simplifies computations by reducing complex problems into manageable subproblems, commonly applied in stochastic programming and optimal transport to model uncertainties and optimize resource allocation effectively.

Advanced Mathematical Frameworks

A. Polyhedral Combinatorics

The study of polyhedral combinatorics involves understanding the facets and vertices of polyhedra associated with combinatorial problems. For instance, the convex hull of all feasible solutions to an ILP can be characterized by its facets, which are crucial for deriving cutting planes in branch-and-cut algorithms.

B. Markov Decision Processes with Combinatorial Structures

Extending MDPs to combinatorial state spaces involves defining transition probabilities and reward functions that respect the combinatorial nature of the states. This requires measure-theoretic rigor to ensure that the stochastic processes are well-defined.

$$V(s) = \max_{a \in \mathcal{A}} \left\{ R(s,a) + \gamma \sum_{s' \in \mathcal{S}} P(s'|s,a)V(s') \right\}$$

C. Functional Analysis in Optimization

Functional analysis provides a framework for analyzing optimization algorithms in infinite-dimensional spaces. For example, considering the optimization problem in a Hilbert space allows the use of inner product properties to study convergence.

$$\|T(x) - T(y)\| \leq L\|x - y\|$$

where T is a Lipschitz continuous operator with Lipschitz constant L.

D. Measure Theory in Stochastic Optimization

Measure theory contains the rigorous treatment of probability measures in stochastic optimization. It facilitates the decomposition of expectation operators and the handling of integrals over complex probability spaces.

$$\mathbb{E}[X] = \int_{\Omega} X(\omega)\, d\mathbb{P}(\omega)$$

Expected Contributions

1. **Theoretical Advancements**:
   - Development of novel polyhedral combinatorics frameworks for tighter ILP formulations.
   - Enhanced stochastic programming models with rigorous measure-theoretic foundations.
   - Advanced convergence analysis of optimization algorithms using functional analytic techniques.

2. **Practical Applications**:
   - Scalable Algorithms for Dynamic Resource Allocation in Telecommunications and Transportation Networks.
   - Robust optimization models for supply chain management under uncertainty.
   - Implementation of combinatorial MDPs in autonomous systems and robotics.
3. **Broader Impact**:
   - Bridging the gaps between combinatorial optimization and advanced mathematical theories.
   - Contributing to the academic community through high-impact publications and collaborative research.
   - Providing robust and scalable solutions applicable across diverse industries.

Methodological Framework

1. Theoretical Development

- **Polyhedral Analysis**: Utilize advanced facet enumeration algorithms to explore the polyhedral structures of combinatorial problems, aiming to derive new cutting planes and strengthen ILP relaxations.
- **Measure-Theoretic Stochastic Models**: Formulate stochastic optimization models within a measure-theoretic framework to ensure rigorous treatment of uncertainty and scenario analysis.
- **Functional Analytic Techniques**: Apply fixed-point theorems and spectral analysis within Banach and Hilbert spaces to study the convergence properties of iterative optimization algorithms.

2. Algorithm Design and Implementation

- **Branch-and-Cut Algorithms**: Develop and enhance branch-and-cut algorithms incorporating newly derived cutting planes from polyhedral combinatorics.
- **Dynamic Programming Extensions**: Extend dynamic programming techniques to handle combinatorial state spaces using measure-theoretic foundations.
- **Operator-Theoretic Optimization**: Design optimization algorithms based on operator theory, ensuring stability and convergence in high-dimensional spaces.

3. Computational Experiments and Validation

- **Simulation Studies**: Conduct extensive simulations on benchmark combinatorial optimization problems to validate the efficacy of proposed algorithms.
- **Real-World Applications**: Apply developed models and algorithms to real-world scenarios in logistics, energy management, and autonomous systems to demonstrate practical utility.

Timeline

1. **Year 1**:
   - Conduct detailed literature review on combinatorial optimization, stochastic programming, and functional analysis applications in OR.
   - Learn about foundational mathematical models integrating polyhedral combinatorics with measure-theoretic probability.
2. **Year 2**:
   - Design and implement branch-and-cut algorithms enhanced with new cutting planes.
   - Formulate and analyze combinatorial MDPs within a measure-theoretic framework.
3. **Year 3**:
   - Conduct computational experiments and simulation studies to validate theoretical models.
   - Develop operator-theoretic optimization algorithms and analyze their convergence properties.
4. **Year 4**:
   - Apply developed methodologies to real-world case studies in logistics and energy management.
   - Publish research findings in high-impact journals and present at international conferences.
   - Collaborate with industry partners to explore applied research opportunities.
5. **Year 5**:
   - Refine and optimize algorithms based on feedback from real-world applications and peer reviews.
   - Develop comprehensive software tools or libraries that implement the proposed optimization algorithms for greater accessibility.
   - Finalize the dissertation, with all research findings, and prepare for defense.
   - Continue publishing in top-tier journals and fostering collaborations for postdoctoral research opportunities.

Conclusion

Therefore, I want to explore the intersection in combinatorial optimization and probability theory and the applications of randomness and navigating chaos in solving the real-world problems. I have been extensively working on improving my foundational knowledge in advanced mathematical topics like probability theory, measure theory, real analysis, functional analysis, and abstract algebra to prepare myself for the grad level courses and research. I plan to continue my research along the way by reaching out to the professors, postdocs, and graduate students working on similar fields. Through this proposal, I hope to outline a vision that resonates my interests for problem solving using mathematics.